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Passive Scalar Advection in Burgers Turbulence: Mapping-Closure Model

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Passive scalar advection in Burgers turbulence is considered. Mapping-closure model based on the amplitudes is used. Advective stretching of scales is included. The simpler nature of this problem affords a convenient framework to clarify some aspects of mapping-closure analysis of coupled stochastic fields. The ability (or lack of it) of the joint action of advective stretching and molecular diffusion to generate non-Gaussian scalar statistics in Burgers turbulence in the absence of a non-Gaussian forcing is investigated.

KEY WORDS: Turbulence; passive scalar.

1. INTRODUCTION

Considerable studies have been made on the advection of a scalar field $\theta(\mathbf{x}, t)$ by a stochastic velocity field $\mathbf{v}(\mathbf{x}, t)$ (see Sreenivasan and Antonia, 1997; Warhaft, 2000; Shariman and Siggia, 2000 for recent reviews). Substantially non-Gaussian statistics can arise for θ even when $\mathbf{v}$ is Gaussian (Kraichnan, 1994).[2] One cause of this different behavior has to do with the relative size of the dominant spatial scales of scalar and velocity fields. If the scale of the advecting field is very small compared with that of the fluctuations in the scalar field, its effect on the latter will be like that of an enhancement of molecular diffusivity. But, if the scales are similar, then possibilities of inducing non-Gaussian statistics in the scalar field can arise.

Advective stretching leads to scalar field being drawn out into thin sheets. However, if the advective stretching or the molecular diffusion acts alone, the scalar and the scalar-gradient at a point remain statistically independent and the one-point statistics of the scalar field described by the probability density function (PDF) $P(\theta, x, t)$ remains Gaussian. On the other hand, if the advective stretching and molecular diffusion act simultaneously, regions that have been highly stretched

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[2] Falkovich *et al.*, (2001) pointed out that the multi-point statistics of the advected scalar fields are closely linked to the collective behavior of the separating Lagrangian trajectories.

experience typically stronger diffusion than other regions. The scalar fluctuations in the highly stretched regions thereby decay rapidly. This selective rapid scalar decay induces a statistical dependence between the scalar and the scalar-gradient at a point. So, in order to calculate a single-point scalar PDF, information concerning the joint statistics of the scalar and its gradient are needed. This necessitates making additional closure assumptions.

In order to deal with such difficulties, Kraichnan and coworkers (Kimura and Kraichnan, 1993; Chen *et al.*, 1989; Kraichnan, 1990) advanced the mapping-closure method to determine PDFs of various quantities in homogeneous turbulence. This method involves mapping a real stochastic field ψ to a Gaussian reference field ψ_0 at each instant t. The evolution equation for the mapping relation

$$\psi = X(\psi_0, t) \tag{1}$$

is to be derived exactly from the evolution for ψ so that the mapping relation (1) may inherit the physics of the evolution in question. Knowledge of the mapping relation (1) would then enable determination of the PDF of ψ as follows

$$P(\psi, t) = P_0(\psi_0)\left(\frac{\partial X}{\partial \psi_0}\right)^{-1} \tag{2}$$

where $P_0(\psi_0)$ is a Gaussian PDF.

Applications of mapping-closure to statistics of velocity field governed by Navier-Stokes dynamics and scalar field driven by such a velocity field are complicated by problems associated with representation of the pressure field as well as the divergence-free condition on the velocity field. Consequently, a full analytic approximation is still forthcoming. To date, only phenomenological approximations exist, which have however yielded excellent fits to the PDF data for velocity gradients (Kraichnan, 1990) and scalar gradients (Shivamoggi, 1995a).

Gao (1991a) considered the PDF of a passive scalar diffusing in homogeneous turbulence by using the mapping-closure method and gave an exact analytical solution for the mapping equation when the advective stretching of scales is ignored. This solution yields scalar PDFs that relax to Gaussian distribution. Pope (1991) generalized the mapping to consider a time-independent Jacobian of transformation while Girimaji (1992) considered evolution from more general reference fields.

The question of the effect of advective stretching of scales on the three-dimensional scalar statistics has not been addressed and is a difficult one to resolve. In this context, some useful insights can be obtained by considering the one-dimensional problem of scalar advection in Burgers turbulence though the ensuing results are not readily applicable to the three-dimensional scalar-advection problem (because there are significant differences between the two problems[3]). On the other

[3] In the absence of diffusivity, three-dimensional advection by a solenoidal velocity field conserves the total amount of the scalar, but the one-dimensional advection described by Eq (4) below does not.

hand, scalar advection in Burgers turbulence appears also to be an interesting problem in its own right (Chertkov *et al.*, 1997). Besides, the simpler nature of this problem (compared with scalar advection by a three-dimensional velocity field) affords a convenient framework to clarify some aspects of mapping-closure analysis of coupled stochastic fields—a topic which has not yet been extensively explored.

2. PASSIVE SCALAR IN BURGERS TURBULENCE

This problem is governed by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{3}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} = \kappa\frac{\partial^2 \theta}{\partial x^2} \tag{4}$$

where ν is the viscosity and κ is the diffusivity.

Equations (3) and (4) show that the velocity gradient $\partial u/\partial x$ and the scalar gradient $\partial\theta/\partial x$ obey the following equations:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\frac{\partial u}{\partial x} = -\left(\frac{\partial u}{\partial x}\right)^2 + \nu\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\right) \tag{5}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\frac{\partial \theta}{\partial x} = -\frac{\partial u}{\partial x}\frac{\partial \theta}{\partial x} + \kappa\frac{\partial^2}{\partial x^2}\left(\frac{\partial \theta}{\partial x}\right). \tag{6}$$

The self-straining term on the right-hand side of Eq (5) leads to a steepening of negative velocity gradient and, therefore, the formation of sawtooth waves with shock fronts whose steepness is limited by the viscous term on the right-hand of Eq (5). During this process, an initially Gaussian velocity field retains a nearly Gaussian univariate distribution $P(u)$ while the univariate distribution $Q(\xi)$ for the velocity gradient $\xi = \frac{\partial u}{\partial x}$ becomes highly intermittent (Gotoh and Kraichnan, 1993, 1998) even at low Reynolds numbers.[4]

Equation (4) shows that the evolution of the scalar is based on the competition between the diffusive relaxation and the advection processes. Equation (4) also

[4] Burgers turbulence driven by a random force $f(x, t)$ has been studied extensively (Bouchaud *et al.*, 1995; Polyakov, 1995; Chekhlov and Yakhot, 1995a,b; Gurarie and Migdal, 1996; Yakhot and Chekhlov, 1996; Ivashkevich, 1997; Balkovsky *et al.*, 1997; Boldyrev, 1997, 1998) because of the possibility that this model might play a role in the development of turbulence theory similar to that played by the two-dimensional Ising model in developing the theory of critical phenomena. For the case of a white-in-time random force, this model has been shown (Chekhlov and Yakhot, 1995a,b) to exhibit statistical properties similar to those of the classical three-dimensional turbulence such as the Kolmogorov energy spectrum. On the other hand, intermittency in this system is caused by the strong shocks underlying the concomitant dynamics. The multi-fractal formulations of compressible turbulence (Shivamoggi, 1995b,c, 1997) fully reproduce, in the ultimate compressibility limit, these main features of Burgers turbulence.

shows that the scalar is conserved along the Lagrangian trajectories in the absence of molecular diffusion. Consequently, the local maxima of the scalar field do not grow in the absence of pumping (Chertkov *et al.*, 1997). This would imply that the statistics of the scalar are likely to remain Gaussian.

The convective stretching term on the right-hand side of equation (6) leads to a steepening of negative scalar gradient when it is in the same direction as that of the velocity gradient. This provides a mechanism for drawing the scalar field into thin sheets in Burgers turbulence.[5]

On the other hand, Eq (6) can be written in the source-free form-

$$\frac{\partial}{\partial t}\left(\frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial \theta}{\partial x}\right) = \kappa\frac{\partial^2}{\partial x^2}\left(\frac{\partial \theta}{\partial x}\right) \tag{7}$$

which implies that the scalar gradient, unlike the scalar, is globally conserved in the absence of molecular diffusion. Therefore, the local maxima of the scalar gradient can grow even in the absence of pumping and possibilities of generation of non-Gaussian statistics for the scalar-gradient PDF exist.

3. THE AMPLITUDE MAPPING CLOSURE MODEL

The mapping-closure scheme is based on the premise that Gaussian reference fields can be distorted into dynamically evolving non-Gaussian fields for the velocity and scalar. Thus, the velocity and scalar fields are taken to evolve according to the amplitude mapping

$$u(x, t) = \mathcal{U}[u_0(z), t] \tag{8}$$

$$\theta(x, t) = \Theta[u_0(z), \theta_0(z), t] \tag{9}$$

where the advective stretching is modeled through a nonlinear transformation between the reference-field scale z and the evolving-field scale x given by

$$\frac{dz}{dx} = J[\xi_0(z), t] \tag{10}$$

and

$$\xi_0 \equiv \frac{\partial u_0}{\partial z}.$$

In the mapping described by Eq (8), the stretching function z has been taken to be the same for both the velocity- and the scalar-fields.[6] Further, the effective

[5] Scalar advection by Burgers turbulence has an interesting feature that follows from the fact that one solution of Eqs (3) and (4), for Prandtl number unity ($\kappa = \nu$), is $u = \theta$. In this case, both the velocity- and the scalar-fields have Gaussian single-point PDF! So, the development of non-Gaussian scalar statistics in Burgers turbulence is predicated on the Prandtl number being non-unity.

[6] Though this step implies forcing a very special relation between the velocity- and the scalar-gradients, it may be mentioned that this step has been found (Shivamoggi, 1995a) to be compatible in generating reliable scalar-gradient statistics in the three-dimensional scalar-advection problem.

intensification ratio J is assumed to depend only on ξ_0 and not on $\eta_0 \left(\equiv \frac{\partial\theta_0}{\partial z}\right)$, which is motivated by the basic premise that the advective stretching is caused by the velocity gradient. The formulations in the following involve J only through some kind of averages (in the coefficients d_i, see (14) below). Therefore, we will not consider explicitly an equation for J, which would fully incorporate the special circumstance that u satisfies Burgers' equation, namely, Eq (3).

Equations (8) and (9) determine the statistics of u and θ via the mapping of Gaussian reference fields $u_0(z)$ and $\theta_0(z)$ at each (z, t), and describe two kinds of nonlinear distortions underlying the mapping closure:

- transformations of amplitudes (the functions $\mathcal{U}$ and Θ),
- change of measure associated with advective squeezing and stretching of the scale z to give x.

The amplitudes $\mathcal{U}$, Θ, and the Jacobian J of the coordinate transformation from z to x are all non-stochastic functions that are determined at each point in the z-space (in which the reference fields live) by local properties — u_0, θ_0, ξ_0 and η_0. The velocity gradient $\xi = \partial u/\partial x$ at each x is statistically independent of velocity u at that point. We consider nonlinear dependence of Θ and θ_0 as well as nonlinear mapping of the z-space to investigate if they have the potential of generating non-Gaussian one-point scalar statistics in the presence of selective rapid decay of scalar in regions that have been strongly strained.[7]

We obtain from (8) and (9),

$$\xi(x,t) \equiv \frac{\partial u}{\partial x} = \frac{\partial \mathcal{U}(u_0,t)}{\partial u_0}\xi_0 J(\xi_0,t) \equiv U(u_0,\xi_0,t) \tag{11}$$

$$\begin{aligned}\eta(x,t) \equiv \frac{\partial \theta}{\partial x} &= \frac{\partial \Theta(u_0,\theta_0,t)}{\partial \theta_0}\eta_0 J(\xi_0,t) + \frac{\partial \Theta(u_0,\theta_0,t)}{\partial u_0}\xi_0 J(\xi_0,t) \\ &\equiv S(u_0,\theta_0,\xi_0,\eta_0,t).\end{aligned} \tag{12}$$

(10) implies that the relation of the joint PDF for (u, θ, ξ, η) to that for $(u_0, \theta_0, \xi_0, \eta_0)$ can be written as

$$P(u,\theta,\xi,\eta,t) = P_0(u_0)P_0(\xi_0)P_0(\theta_0)P_0(\eta_0)\left\{\frac{\partial\mathcal{U}}{\partial u_0}\frac{\partial U}{\partial \xi_0}\frac{\partial \Theta}{\partial \theta_0}\frac{\partial S}{\partial \eta_0}\right\}^{-1}\frac{N(t)}{[J(\xi_0,t)]^2} \tag{13}$$

where $N(t)$ is a normalization factor

$$N(t) \equiv \left\{\int \frac{P_0(\xi_0)}{[J(\xi_0,t)]^2}d\xi_0\right\}^{-1}.$$

[7] Linear dependence of Θ on θ_0, in the presence of advective stretching, on the other hand, is able to generate non-Gaussian statistics in the scalar-gradient field (see Appendix). Linear dependence of $\mathcal{U}$ on u_0 is also able to generate non-Gaussian statistics in the velocity-gradient field (Kraichnan, 1990).

Differentiation of (10) yields

$$\frac{\partial^2\theta}{\partial x^2} = \left(J^2\frac{\partial\eta_0}{\partial z} + \eta_0 J\frac{\partial J}{\partial\xi_0}\frac{\partial\xi_0}{\partial z}\right)\frac{\partial\Theta}{\partial\theta_0} + \eta_0^2 J^2\frac{\partial^2\Theta}{\partial\theta_0^2} + \left(J^2 + \xi_0 J\frac{\partial J}{\partial\xi_0}\right)\frac{\partial\xi_0}{\partial z}\frac{\partial\Theta}{\partial u_0} + \xi_0^2 J^2\frac{\partial^2\Theta}{\partial u_0^2} + 2\xi_0\eta_0 J^2\frac{\partial^2\Theta}{\partial\theta_0\partial u_0}.$$

Using the Gaussian relations

$$\left\langle\frac{\partial\xi_0}{\partial z}\middle| u_0\right\rangle = -\frac{\langle\xi_0^2\rangle}{\langle u_0^2\rangle}u_0 \tag{14}$$

$$\left\langle\frac{\partial\eta_0}{\partial z}\middle| \theta_0\right\rangle = -\frac{\langle\eta_0^2\rangle}{\langle\theta_0^2\rangle}\theta_0 \tag{15}$$

where $\langle\cdot|\psi_0\rangle$ denotes the ensemble mean conditional on given value of ψ_0 at (z, t), we have from (12)

$$\left\langle\frac{\partial^2\theta}{\partial x^2}\,|u_0, \theta_0\right\rangle = -[d_1(t)\theta_0 + d_2(t)u_0]\frac{\partial\Theta}{\partial\theta_0} + d_3(t)\frac{\partial^2\Theta}{\partial\theta_0^2} - d_4(t)u_0\frac{\partial\Theta}{\partial u_0} + d_5(t)\frac{\partial^2\Theta}{\partial u_0^2} + d_6(t)\frac{\partial^2\Theta}{\partial\theta_0\partial u_0} \tag{16}$$

where

$$\begin{aligned}
d_1(t) &\equiv \frac{\langle\eta_0^2\rangle}{\langle\theta_0^2\rangle}N(t)\\
d_2(t) &\equiv \frac{\langle\xi_0^2\rangle}{\langle u_0^2\rangle}\int \eta_0 J\frac{\partial J}{\partial\xi_0}P(\xi_0)P(\eta_0)\frac{N}{J^2}d\xi_0 d\eta_0 = 0\\
d_3(t) &\equiv \langle\eta_0^2\rangle N(t)\\
d_4(t) &\equiv \frac{\langle\xi_0^2\rangle}{\langle u_0^2\rangle}N(t)\left(1 + \int \xi_0 P_0(\xi_0)\frac{1}{J}\frac{\partial J}{\partial\xi_0}d\xi_0\right)\\
d_5(t) &\equiv \langle\xi_0^2\rangle N(t)\\
d_6(t) &\equiv 2\int \xi_0\eta_0 J^2 P_0(\xi_0)P_0(\eta_0)\frac{N}{J^2}d\xi_0 d\eta_0 = 0.
\end{aligned} \tag{17}$$

Using (8), (9), and (14), Eq (4) leads to

$$\frac{\partial\Theta}{\partial t} = -\kappa d_1(t)\theta_0\frac{\partial\Theta}{\partial\theta_0} + \kappa d_3(t)\frac{\partial^2\Theta}{\partial\theta_0^2} - \kappa d_4(t)u_0\frac{\partial\Theta}{\partial u_0} + \kappa d_5(t)\frac{\partial^2\Theta}{\partial u_0^2}. \tag{18}$$

The effect of advective stretching is contained in the parameter $d_4(t)$. Note that Eq (15) admits a solution for which the mapping function Θ is a linear function of θ_0 and u_0 signifying a Gaussian statistics for θ. It is pertinent to ask if other solutions tend to this solution in some asymptotic limit.

In order to facilitate finding these solutions, we introduce the transformation

$$(u_0, \theta_0, t) \to (\zeta(u_0, \theta_0, t), \tau(t)) \tag{19}$$

where

$$\zeta(t) \equiv u_0 T_1(t) + \theta_0 T_2(t). \tag{20}$$

Equation (15) then becomes

$$\begin{aligned} \frac{\partial \Theta}{\partial \tau}\tau' + (u_0 T_1' + \theta_0 T_2')\frac{\partial \Theta}{\partial \zeta} &= -\kappa d_1 \theta_0 T_2 \frac{\partial \Theta}{\partial \zeta} - \kappa d_4 u_0 T_1 \frac{\partial \Theta}{\partial \zeta} \\ &\quad + \kappa\left(d_3 T_2^2 + d_5 T_1^2\right)\frac{\partial^2 \Theta}{\partial \zeta^2}. \end{aligned} \tag{21}$$

The solutions of Eq (17) can be obtained by transforming it into either diffusion equation (Gao, 1991b) or Hermite's equation (Gao and O'Brien, 1991) (see also Takaoka, 1999 who developed the corresponding formulation for the velocity field evolving according to Eq (3)).

3.1. Reduction to Diffusion Equation

We choose here, therefore,

$$\begin{aligned} \tau' - \kappa\left(d_3 T_2^2 + d_5 T_1^2\right) &= 0 \\ T_2' + \kappa d_1 T_2 &= 0 \\ T_1' + \kappa d_4 T_1 &= 0. \end{aligned} \tag{22}$$

Equations (18) can be solved formally to give

$$T_2(t) = e^{-\kappa \int_0^t d_1(t')dt'} \tag{23}$$

$$T_1(t) = e^{-\kappa \int_0^t d_4(t')dt'} \tag{24}$$

$$\tau(t) = \kappa \int_0^t \left[d_3(t')T_2^2(t') + d_5(t')T_1^2(t')\right]dt'. \tag{25}$$

Using (18), Eq (17) reduces to diffusion equation

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial \zeta^2}, \tag{26}$$

the solution of which is given by

$$\Theta(\zeta,\tau)=\frac{1}{2\sqrt{\pi\tau(t)}}\int_{-\infty}^{\infty}\Theta(\beta,0)e^{-\frac{[\beta-\zeta(u_0,\theta_0,t)]^2}{4\tau(t)}}d\beta. \tag{27}$$

Noting from (19) that

$$t=0: T_1=1,\quad T_2=1\quad \text{and}\quad \tau=0 \tag{28}$$

we have

$$t=0:\Theta=\Theta(\zeta_0,0), \tag{29}$$

where

$$\zeta_0\equiv u_0+\theta_0.$$

Following Pope (1985), and recalling (22) and (23), consider the initial condition

$$\Theta(\zeta_0,0)=\begin{cases}-1, & \zeta_0<0\\ 1, & \zeta_0>0.\end{cases} \tag{30}$$

(21) then becomes

$$\Theta(\zeta,\tau)=erf\left[\frac{\zeta(t)}{2\sqrt{\tau(t)}}\right]. \tag{31}$$

Now, taking $P_0(\zeta_0)$ to be Gaussian

$$P_0(\zeta_0)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\zeta_0^2}{2\sigma^2}} \tag{32}$$

the relation

$$P(\theta,t)\frac{\partial\theta}{\partial\zeta}\frac{\partial\zeta}{\partial\zeta_0}=P_0(\zeta_0) \tag{33}$$

then shows that θ remains Gaussian even in the presence of advective stretching.

3.2. Reduction to Hermite's Equation

We choose here, therefore,

$$\begin{aligned}\tau'-\kappa\left(d_3T_2^2+d_5T_1^2\right)&=0\\ T_2'+\kappa d_1T_2-\kappa\left(d_3T_2^2+d_5T_1^2\right)T_2&=0\\ T_1'+\kappa d_4T_1-\kappa\left(d_3T_2^2+d_5T_1^2\right)T_1&=0.\end{aligned}$$

The first of equations (28) may be formally solved to give

$$\tau(t)=\kappa\int_0^t\left[d_3(t')T_2^2(t')+d_5(t')T_1^2(t')\right]dt'. \tag{34}$$

Exact solutions for T_1 and T_2 have not been found.

Using (28), Eq (17) reduces to Hermite's equation

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial \zeta^2} - \zeta \frac{\partial \Theta}{\partial \zeta} \tag{35}$$

the solution of which is given by

$$\Theta(\zeta, \tau) = \sum_{n=0}^{\infty} \frac{e^{-n\tau(t)}}{\sqrt{2\pi}\ n!} \left(\int e^{-\beta^2/2} H_n(\beta)\Theta(\beta, 0) d\beta \right) H_n(\zeta(u_0, \theta_0, t)), \tag{36}$$

where $H_n(x)$ is the nth-order Hermite function

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}). \tag{37}$$

(31) shows that we have for large t (or large τ, from (29))

$$\Theta \sim \zeta e^{-\tau} \tag{38}$$

so that θ relaxes to Gaussian even in the presence of advective stretching.

4. DISCUSSION

In this paper, we have investigated the effect of advective stretching of scales on the scalar statistics. To facilitate analysis, we have considered scalar advection in Burgers turbulence governed by Eqs (3) and (4). Writing Eq (4) in the scalar-conserving form

$$\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial x}(u\theta) = \kappa \frac{\partial^2 \theta}{\partial x^2} + \theta \frac{\partial u}{\partial x} \tag{39}$$

one notices that non-Gaussianity of θ can arise from one more of the following aspects:

- the source term $\theta \frac{\partial u}{\partial x}$, which is a compressibility effect;
- the advective stretching of scales,
- the introduction of a non-Gaussian forcing into the system.

The amplitude-mapping-closure formulations in Section 3 show that neither the compressibility effect nor the advective stretching scales is able to generate non-Gaussian scalar statistics in Burgers turbulence. So, the introduction of a non-Gaussian forcing appears to be the only way to generate non-Gaussian scalar statistics in Burgers turbulence in the amplitude-mapping-closure model. Similar results have been given previously with other closure schemes (Chertkov *et al.*, 1997).

APPENDIX

Linear dependence of Θ and θ_0 (and $\mathcal{U}$ on u_0) in the presence of advective stretching turns out to be able to generate non-Gaussian statistics in the scalar-gradient field.

Suppose that the velocity and scalar fields evolve according to the linear maps:

$$u(x,t) = c_1(t)u_0(z) \tag{A.1}$$

$$\theta(x,t) = c_2(t)\theta_0(z) \tag{A.2}$$

with

$$\frac{dz}{dx} = J[\xi_0(z), t]. \tag{A.3}$$

We obtain from (A.1) and (A.2),

$$\xi = c_1(t)J(\xi_0, t)\xi_0 \tag{A.4}$$

$$\eta = c_2(t)J(\xi_0, t)\eta_0. \tag{A.5}$$

Equations (5) and (6) then imply the following evolution equations for the stretching function $J(\xi_0, t)$:

$$\frac{\partial J}{\partial t} = -c_1|\xi_0|J^2 - \nu k_d^2 J^3 \tag{A.6}$$

$$\frac{\partial J}{\partial t} = -c_1|\xi_0|J^2 - \kappa \hat{k}_d^2 J^3 \tag{A.7}$$

where

$$k_d^2 \equiv \frac{\left\langle \left(\frac{d\xi_0}{dz}\right)^2 \right\rangle}{\langle \xi_0^2 \rangle}, \quad \hat{k}_d^2 \equiv \frac{\left\langle \left(\frac{d\eta_0}{dz}\right)^2 \right\rangle}{\langle \eta_0^2 \rangle}.$$

The consistency between Eqs (A.4) and (A.5) requires that the Prandtl number must satisfy

$$\frac{\nu}{\kappa} = \frac{\hat{k}_d^2}{k_d^2} \tag{A.8}$$

which is a compatibility condition for a linear map for scalar advection in Burgers turbulence.

We then have from Eqs (A.4) and (A.5) in the stationary state, on taking $c_1(t) = 1$,

$$J = \frac{|\xi_0|}{\nu k_d^2}. \tag{A.9}$$

Using (A.7), we obtain from (A.3b)

$$|\eta| = \frac{|\xi_0||\eta_0|}{\nu k_d^2}. \tag{A.10}$$

If ξ_0 and η_0 are assumed to be multi-variate Gaussian fields given by the PDF:

$$P(\xi_0, \eta_0) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{\xi_0^2}{\sigma_1^2} - \frac{2\rho\xi_0\eta_0}{\sigma_1\sigma_2} + \frac{\eta_0^2}{\sigma_2^2}\right]} \tag{A.11}$$

where

$$\sigma_1^2 \equiv \langle\xi_0^2\rangle, \quad \sigma_2^2 \equiv \langle\eta_0^2\rangle, \quad \rho \equiv \frac{\langle\xi_0\eta_0\rangle}{\sigma_1\sigma_2},$$

then, following the development in Shivamoggi (1995a), (A.8) leads to the following non-Gaussian PDF for the scalar gradient:

$$P(\eta) = \frac{\nu k_d^2}{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\frac{\rho|\eta|}{\sigma_1\sigma_2(1-\rho^2)/\nu k_d^2}} K_0\left(\frac{|\eta|}{\sigma_1\sigma_2(1-\rho^2)/\nu k_d^2}\right) \tag{A.12}$$

$K_0(x)$ being the modified Bessel function of the second kind.

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